8. CHEREPANOV G.P., Mechanics of Brittle Fracture. Nauka, Moscow, 1974.
9. WILLIAMS M.L., Stress singularities resulting from various boundary conditions in angular corners of plates in extension, J. Appl. Mech. 19, 4, 1952.
10. UFLYAND YA.S., Integral Transformations in Problems of the Theory of Elasticity. Nauka, Leningrad, 1967.
11. KIPNIS L.A., The homogeneous problem for a wedge with a symmetric crack at the tip. PMM, 45, 2, 1981.
12. NOBLE B., Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations. Pergamon Press, 1958.

Translated by L.K.

PMM U.S.S.R.,Vol.53,No.1,pp.133-135,1989
Printed in Great Britain
0021-8928/89 $10.00+0.00

ON A KELVIN PROBLEM*

V.V. KOZLOV

A problem of the stability of equilibrium of a system of interacting particles distributed within a bounded volume of Euclidean space is considered. Sufficient conditions for the instability and existence of the motions approaching the position of equilibrium without bounds, containing the Kelvin theorem /1/ as a special case, are obtained. The results are based on the general theory of instability of equilibrium in a force field with a subharmonic force function.

1. Let us consider the dynamics of a reversible system with kinetic energy $T = (g_{ij}v^iv^j)/2$ and force function $U(x)$. The motions are described by the Lagrange equations

$$(L'_{v^i})^{\cdot} - L'_{x^i} = 0, \quad v^i = d(x^i)/dt, \quad L = T + U, \quad i \leqslant n \tag{1.1}$$

The coefficients of the metric tensor g_{ij} and the function U are assumed to depend continuously on the x coordinates. We assume that the point $x = 0$ is critical for the force function U, and therefore $x = 0$ will represent the equilibrium of the system (1.1). We can assume that $U(0) = 0$. The function U will be called subharmonic if $\Delta U \geqslant 0$ where Δ is a Laplace-Beltrami operator taken with the minus sign:

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,g^{ij}\frac{\partial}{\partial x^j}\right), \quad g = \det\|g_{ij}\|$$

It is clear that the condition of subharmonicity of the force function does not depend on the choice of the Lagrangian coordinates x^i.

Theorem 1. Let us assume that the force function U is subharmonic and its Maclaurin's series is different from zero. Then the equilibrium $x = 0$ will be unstable. In the analytic case the condition of subharmonicity is sufficient for the instability to occur.

Proof. Let g_0^{ij} be the values of the metric tensor at the point $x = 0$. We expand the force function U in a series in terms of homogeneous forms: $U_m + U_{m+1} + \ldots$, $m \geqslant 2$. It can be confirmed that $\Delta U = \Delta_0 U_m + \ldots$ where Δ_0 is the Laplace-Beltrami operator of the metric g_0^{ij} and repeated dots denote terms of order $\geqslant m - 2$. Since $\Delta U \geqslant 0$, we have $\Delta_0 U_m \geqslant 0$. The coefficients of the operator Δ_0 are independent of x, therefore the function U_m is subharmonic in the sense of the classical definition /2/.

Using the well-known inequality

**Prikl.Matem.Mekhan.*,53,1,165-167,1989

$$0 = U_m(0) \leqslant \frac{1}{s_n r^{n-1}} \int_S U_m d\sigma$$

where S is a sphere of radius r with centre at the point $x = 0$, s_n is the surface of the unit sphere, we find that the form U_m must take positive values. Therefore U_m has no maximum at the point $x = 0$. We proved in /3/ that under this condition solutions of (1.1) exist, which approach the point $x = 0$ without bounds as $t \to +\infty$ or $t \to -\infty$. This in turn implies the instability of the equilibrium $x = 0$.

Corollary 1. Let the coefficients of the metric tensor g_{ij} be analytic, and the force function harmonic: $\Delta U = 0$. Then any equilibrium will be unstable.

Indeed, if the Maclaurin's series U is non-zero, the conclusion of the corollary will follow from Theorem 1. Otherwise $U \equiv 0$, since the harmonic functions are analytic. Every point $x = x_0$ will be a neutral equilibrium, and all of them are, of course, unstable.

From Corollary 1 we conclude, in particular, that the well-known Earnshaw hypothesis on the instability of the equilibrium of a system of free charges in a stationary electric field in three-dimensional space /1, 4/, holds. The hypothesis was justified earlier for the most important special case in which $U = U_2 + U_3 + \ldots$ and $U_2 \not\equiv 0$ /5/.

2. The problem of the stability of the equilibrium of a system of mutually repelling material points confined within a bounded volume, was studied by Kelvin. Some of these points may lie on the boundary, and the exact formulation of the problem must be based on the theory of selfreleasing constraints. First we shall consider the conditions of stability of the system of mutually interacting points in an n-dimensional Euclidean space, where a number of these points is at rest. From the point of view of practical applications, the most interesting case is that if $n \leqslant 3$. Let U_{ij} be the force function of the interacting particles with masses m_i, and m_j $(i \neq j)$. The function depends only on their mutual distances.

Theorem 2. Let us assume that the functions $U_{ij}(r)$ are analytic for $r > 0$ $(i \neq j)$ and

$$\frac{d}{dr}\left(r^{n-1}\frac{dU_{ij}}{dr}\right) \geqslant 0 \tag{2.1}$$

Then any equilibrium will be unstable.

Proof. Let $x_i^1, \ldots, x_i^n$ be the Cartesian coordinates of the point of mass m_i. Then $T = \Sigma m_i (v_i^k)^2/2$. The corresponding differential operator Λ will have the form

$$\sum \frac{1}{m_i}\left(\frac{\partial^2}{\partial (x_i^1)^2} + \cdots + \frac{\partial^2}{\partial (x_i^n)^2}\right)$$

Let $x_j^1, \ldots, x_j^n$ be the coordinates of another point of mass m_j, and $r_{ij} = \left[\sum (x_i^k - x_j^k)^2\right]^{1/2}$ be the distance between them. It is clear that the quantity

$$\sum_k \frac{\partial^2 U_{ij}(r_{ij})}{\partial (x_i^k)^2}$$

is equal to the left-hand side of inequality (2.1). The complete force function of the system of interacting particles is equal to $U = \sum_{i<j} U_{ij}$. Taking into account the inequalities (2.1) we find, that U is a subharmonic function. The instability of the equilibrium now follows from Theorem 1.

We shall consider, as an example, a power law of the interaction $U_{ij}(r) = a_{ij} r^{\alpha}$. In the case of attraction $a_{ij} < 0$, and in case or repulsion $a_{ij} > 0$. From the inequality (2.1) we obtain

$$a_{ij}\,\alpha\,(n + \alpha - 2) \geqslant 0 \tag{2.2}$$

If the points attract (repel) each other, then the equilibrium is unstable when $\alpha(n + \alpha - 2) \leqslant 0$ $(\alpha(n + \alpha - 2) \geqslant 0)$. Then $\alpha = 2 - n$, the force function is harmonic and we again obtain the Earnshaw theorem. In the case of linear forces $\alpha = 2$, and hence the equilibrium of the particles repelling each other elastically will always be unstable (compare with /1/).

In the special case when $2n$ stationary points are distributed over n straight coordinate lines at equal distances from the point $x = 0$ and the coefficients a_{ij} are equal to each other, the inequality (2.2) will serve as the criterion of instability of equilibrium of the particle situated at the point $x = 0$.

Condition (2.1) becomes particularly simple when $n = 1$. If the force function of the dual interaction between the particles on a straight line is concave upwards, any equilibrium will be unstable. In particular, any equilibrium configuration of gravitating points on a straight line will be unstable. Conversely, in the case of repulsion, stable equilibrium configurations are possible. The simplest example of this is the equilibrium of a charge situated between stationary charges of the same sign.

3. We will now consider a more complicated case, when some of the particles in the state of equilibrium are distributed over a closed, smooth regular hypersurface Σ. In the course of the analysis of the stability we shall assume that these particles do not depart from Σ during their motion (i.e. the constraints are not selfreleasing). The dynamics of such a system of particles is again described by the Lagrange Eqs.(1.1), but the metric T will no longer be plane. Let U_{ij} again denote the force functions of dual interaction depending only on the distance between the interacting particles.

If, in the state of equilibrium, not all particles lie on the surface Σ, then a new mechanical system with fewer degrees of freedom can be considered. The system is obtained by fixing the positions of the particles lying on Σ. Let U' be the force function of the new system. It is clear that the configuration of the particles in the initial system represents the equilibrium of the partially "frozen" system.

Theorem 3. Let us assume that not all particles lie on Σ in the state of equilibrium, that the inequalities (2.1) hold, and $U' = U_2' + U_3' + \ldots, U_2' \not\equiv 0$. Then the equilibrium is unstable.

Proof. Let us put $U = U_2 + U_3 + \ldots$. It is clear that the form U_2' represents the form U_2 restricted to the configurational space of the frozen system. Since $U_2' \not\equiv 0$, it follows according to Sect.2 that the form U_2' has no maximum in the state of equilibrium. Therefore the quadratic form U_2 has the same property. The instability of the equilibrium now follows from the Lyapunov theorem /5/.

If not all particles lie on Σ in the state of equilibrium and the inequalities (2.1) hold, then the force function U has no local maximum. The problem of instability however runs, in this case, into the unsolved problem of inverting the Lagrange-Dirichlet theorem.

Corollary 2. (Kelvin's theorem /1/). Assume that the system of particles repelling each other elastically and confined within a bounded volume V, is in equilibrium and that not all particles lie on the boundary $\partial V = \Sigma$. Then the equilibrium is unstable.

Indeed, in this case $U' \equiv U_2'$, and in the case of elastic repulsion the form U_2' is, according to Sect.2, a subharmonic function.

If all interacting particles lie on the surface Σ, then Theorem 1 should be used in determining the conditions of stability.

REFERENCES

1. ROUTH E.G., Dynamics of a System of Solids. Nauka, Moscow, 1983.
2. HAYMAN U.K. and KENNEDY P.B., Subharmonic Functions. London Math. Soc. 1, 1977.
3. KOZLOV V.V., Asymptotic motions and inversion of the Lagrange-Dirichlet theorem. PMM, 50, 6, 1986.
4. TAMM I.E., Fundamentals of the Theory of Electricity. Nauka, Moscow, 1966.
5. LYAPUNOV A.M., The General Problem of the Stability of Motion. Gostekhizdat, Leningrad, 1950.

Translated by L.K.